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Half quantization

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Abstract

A general dynamical system composed of two coupled sectors is considered. The initial-time configuration of one of these sectors is described by a set of classical data while the other is described by standard quantum data. These dynamical systems will be named half quantum and the aim of this paper is to derive their dynamical evolution. The standard approach would be to use quantum mechanics to make predictions for the time evolution of the half quantum initial data. The main problem is how quantum mechanics can be applied to a dynamical system whose initial time configuration is not described by a set of fully quantum data. A solution to this problem is presented and used as a guideline to obtain a general formulation of coupled classical–quantum dynamics. Finally, a quantization prescription mapping a given classical theory to the corresponding half quantum one is presented.

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1. Introduction

Quasiclassical dynamics [1], hybrid dynamics [2] and, in this paper, half quantum mechanics are some of the several attempts [3–7] to obtain a consistent formulation of coupled classical–quantum dynamics. The motivation to develop such a theory comes from a variety of different sources. The theory is expected to make important contributions to clarify the measurement procedure in quantum mechanics, where one would like to obtain an analytic description of the wavefunction collapse [8–10]. Closely related is the problem of developing a consistent quantization procedure for closed dynamical systems [11, 12]. Other important applications are expected. These include semiclassical gravity, quantum field theory in curved spacetime and quantum cosmology [5, 8, 11, 13, 14].

Two main approaches to the problem have been followed: in [1, 3, 4] a set of axioms defining the quasiclassical dynamics were proposed and motivated in terms of the consistency of the resulting theory. On the other hand is the deductive approach where the intention is to derive the coupled classical–quantum dynamics from quantum mechanics [2, 5–7]. In this paper we shall follow this second approach. We assume that, just like classical mechanics,

half quantum mechanics is an approximate description of quantum mechanics that derives its validity from reproducing, 'in some appropriate sense', the predictions of the underlying theory of quantum mechanics.

Our approach will be as follows: a general half quantum system is composed of two coupled sectors. One of these sectors is named classical and the other quantum. The initial data for a half quantum system is given by a set of classical data and associated error margins $(O_i(t=0), \delta_i(t=0))$ for the classical sector observables plus a standard quantum datum, say an initial-time wavefunction $|\phi^{\mathrm{Q}}\rangle$ for the quantum sector. The classical data provide only an approximate description of the true physical configuration of the classical sector. The complete description of this configuration is given by some unknown wavefunction $|\phi^{\mathrm{c}}\rangle$. Clearly not all wavefunctions $|\phi^{\mathrm{c}}\rangle$ provide a description of the classical sector initial time configuration consistent with the classical description $(O_i(t=0), \delta_i(t=0))$. Our first step will be to identify a set of wavefunctions $|\phi^{\mathrm{c}}\rangle$ compatible with the classical initial data. We will be able to do this by using a classicality criterion that was presented in a related paper [15] and which proved to work out successfully when the intention was to study the consistency between the full classical and full quantum descriptions of a general dynamical system.

The next step is to use quantum mechanics to obtain the time evolution of the class of quantum initial data $|\phi\rangle = |\phi^{\mathrm{Q}}\rangle|\phi^{\mathrm{c}}\rangle$. The predictions of quantum mechanics, i.e. the time evolution of the class of wavefunctions $|\phi\rangle$, will not be completely determined. This is so because we do not have a single initial data wavefunction, but instead we are calculating the time evolution for a class of initial data wavefunctions. Therefore, quantum mechanics provides a set of predictions inside an error interval. The main result is then that these predictions might be fully recovered by an appropriate formulation of classical–quantum dynamics, which will be named half quantum mechanics. In this formulation the dynamical system is not fully quantized, the classical data describing the initial-time configuration of the classical sector are explicitly used and dynamics is obtained as the time evolution of the classical and quantum initial data. Still, we are able to recover the predictions of quantum mechanics for the time evolution of the class of wavefunctions $|\phi\rangle$. This is the desired result. It means that the half quantum framework is derived as the appropriate limit of quantum mechanics. We will find that the theory derived here is just the same as postulated by Boucher and Traschen in [4]. The approach however, is rather different. In that paper the theory was motivated by the properties one would like to see satisfied by a theory of coupled classical–quantum dynamics.

Our derivation presents some interesting features:

(i) it explicitly provides the degree of precision of the half quantum predictions.
(ii) It states what type of initial datum and dynamical behaviour a system should have so that it can be described by the half quantum framework.
(iii) It sets out a general procedure to develop other, eventually more consistent or better behaved, classical–quantum dynamics frameworks.
(iv) It provides a half quantization procedure mapping the classical formulation of a given dynamical system to its half quantum formulation.

2. From quantum mechanics to half quantum mechanics

Let us set out the preliminaries: we are given a dynamical system with $N + M$ degrees of freedom. N represents the number of degrees of freedom of the quantum sector, while M concerns the classical sector. The phase space of the classical formulation of the system is spanned by a set of canonical variables $\{q_k, p_k\}, k = 1 \ldots (M+N)$, more succinctly designated by $O_k, k = 1 \ldots 2(M+N)$. The classical sector canonical variables are denoted by

$(q_i, p_i), i = 1 \ldots M$ or just by $O_i, i = 1 \ldots 2M$ and the quantum sector canonical variables by $(q_\alpha, p_\alpha), \alpha = (M+1) \ldots (M+N)$ or $O_\alpha, \alpha = (2M+1) \ldots 2(M+N)$. The total phase space is assumed to have a structure given by $T^*M_1 \otimes T^*M_2$ where T^*M_1 is the classical sector phase space and T^*M_2 is the quantum sector phase space. By performing the Dirac quantization [16, 17] we obtain the quantum formulation of the dynamical system. We also supply a complete set of commuting observables (CSCO). We will take the CSCO to be $\{\hat{q}_i, \hat{q}_\alpha\}$. The set of common eigenvectors of the CSCO spans the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Taking into account the structure of the Hilbert space the general eigenvector might be written as $|k_1, \ldots, k_N\rangle|z_1, \ldots, z_M\rangle$, where the k are eigenvalues of the operators $\hat{q}_\alpha$ and the z are those of the operators $\hat{q}_i$.

The aim now is to use the full quantum formulation of the dynamical system to study the time evolution of the half quantum initial data. This is far from being straightforward, the first problem being how to use the half quantum initial data to produce fully quantum initial data for the quantum theory. This problem will be approached in this section.

2.1. From quantum mechanics to half quantum mechanics—kinematics

The half quantum dynamical system is composed of two sectors. The initial-time configuration of one of these sectors is described by a set of classical data. That is, a value O_i^0 and an error margin δ_i are assigned to each classical sector observable O_i. The aim is to convert these classical data into a fully quantum one, $|\phi^c\rangle \in \mathcal{H}_1$. Clearly, not all wavefunctions $|\phi^c\rangle$ will be suitable. We are looking for a class of wavefunctions $\{|\phi^c\rangle\}$ providing a description of the initial-time configuration consistent with the classical description (O_i^0, δ_i). To obtain this class of wavefunctions we impose that $|\phi^c\rangle$ should satisfy a set of classicality conditions that were defined and studied in [15]. More precisely, we require $|\phi^c\rangle$ to be L-order classical ($L \in \mathcal{N}$) with respect to the classical data (O_i^0, δ_i). The higher the order of classicality L, the greater will be the degree of consistency between the classical and the quantum descriptions. Alternatively, it may be worth thinking about the classicality criterion in an equivalent but slightly different perspective: the degree of classicality can be seen as the degree of *validity* of the classical description of the dynamical system. The classical description is *valid* up to some degree L if the true, physical configuration of the dynamical system is given by a wavefunction $|\phi^c\rangle$, L-order classical with respect to that classical description. Finally, let us point out that we will not fix the value of L. In fact, L is to be one of the parameters of the formalism and later we will find that its value is related to the precision of the half quantum predictions.

Let us give a brief review of the definition of the classicality criterion. Let $O_k(t)$ be the classical time evolution of an arbitrary fundamental observable (belonging to the classical or to the quantum sector) and let S_{i_a} be *any* sequence of classical sector observables $S_{i_a} = O_{i_1}, O_{i_2} \ldots O_{i_n}$—associated with a sequence $1 \leqslant i_a \leqslant 2M$, $a = 1 \ldots n$ (n is arbitrary)—such that

$$\frac{\partial^n O_k(t)}{\partial S_{i_a}} = \frac{\partial^n O_k(t)}{\partial O_{i_1} \ldots \partial O_{i_n}} \neq 0 \tag{1}$$

for some $k = 1 \ldots 2(N+M)$. With all sequences satisfying the former relation we can obtain a set of mixed error kets (the reader should refer to the appendix for the relevant definitions):

$$|E_{S_{i_a}}\rangle = (\hat{O}_{i_1} - O_{i_1}^0)(\hat{O}_{i_2} - O_{i_2}^0) \ldots (\hat{O}_{i_n} - O_{i_n}^0)|\phi^c\rangle \tag{2}$$

where the quantities $O_{i_a}^0$ refer to the values of the corresponding observables O_{i_a} at the initial time. The classical sector initial-time wavefunction $|\phi^c\rangle$ will be *first-order classical* if it satisfies

$$\langle E_{S_{i_a}}|E_{S_{i_a}}\rangle \leqslant (\delta_{S_{i_a}})^2 = \delta_{i1}^2 \delta_{i2}^2 \ldots \delta_{in}^2 \tag{3}$$

for all the sequences S_{i_a} determined in (1). In the former equation δ_{ia} are the error margins associated with the classical initial data. Notice that given the classical initial data and its error margins the former inequalities constitute a set of requirements on the functional form of the wavefunction $|\phi^c\rangle$. To go further we consider the L-order sequences $S^L_{i_a} = S_{i_a} S_{i'a} \dots S_{i''_a}$ constituted by L arbitrary first-order sequences S_{i_a} (determined in (1)) and write the system of inequalities (3) for these sequences. If the wavefunction $|\phi^c\rangle$ satisfies (3) for all possible L-order sequences then we say that $|\phi^c\rangle$ is L-order classical. The set $\{|\phi^c\rangle\}_L$ of all L-order classical wavefunctions is the class of wavefunctions we wanted to determine.

The classicality criterion was developed in a related paper [15]. There the intention was to study the conditions that the initial-time wavefunction $|\phi\rangle$ of a general quantum system (with M degrees of freedom) should satisfy so that its time evolution—given by $|\phi(t)\rangle$, solution of the Schrödinger equation—is consistent with the classical treatment of the system, given by $O_i(t)$ ($i = 1 \dots 2M$), solutions of the Hamiltonian equations and associated error margins $\delta_i(t)$. The main result of that paper was that if $|\phi\rangle$ satisfies the conditions (3) for all sequences $S^L_{i_a}$, formed by L arbitrary fundamental sequences determined in (1) (i.e. if $|\phi\rangle \in \{|\phi\rangle\}_L$), then $|\phi(t)\rangle$ satisfies the following property for all times: in the representation of any of the observables $\hat{O}_i$, the wavefunction $|\phi(t)\rangle$ has at least a probability p confined to the interval $I_i = [O_i(t) - \delta_i(t)/(1-p)^{1/2L}, O_i(t) + \delta_i(t)/(1-p)^{1/2L}]$, that is $\sum_{a_i \in I_i, n} |\langle a^n_i | \phi(t)\rangle|^2 \geqslant p$ for all $i = 1 \dots 2M$ and for all $0 \leqslant p < 1$. In the former expression $|a^n_i\rangle$ is a general eigenvector of the operator $\hat{O}_i$ with associated eigenvalues a_i and degeneracy index n. By simple inspection we notice that the higher the degree of classicality L, the more confined are the probabilistic distribution functions associated with $|\phi(t)\rangle$, in the representation of any of the observables $\hat{O}_i$, around the corresponding classical intervals $[O_i(t) - \delta_i(t),\ O_i(t) + \delta_i(t)]$. Therefore condition (3) provides a suitable measure of classicality and it might be used to determine the set of wavefunctions compatible with the classical initial data.

Let us see how this works in the simple example of the harmonic oscillator. The classical Hamiltonian is given by $H = \frac{1}{2}(q^2 + p^2)$, where q and p are a pair of canonical variables and, to make it simple, we have set $w = m = 1$. By solving the equations of motion we obtain the classical time evolution of the canonical variables and the corresponding error margins:

$$\begin{aligned} q(t) &= q(0)\cos t + p(0)\sin t \qquad & \delta_q(t) &= |\cos t|\delta_q(0) + |\sin t|\delta_p(0) \\ p(t) &= q(0)\sin t + p(0)\cos t \qquad & \delta_p(t) &= |\sin t|\delta_q(0) + |\cos t|\delta_p(0). \end{aligned} \tag{4}$$

Let $|\phi\rangle$ be the initial data wavefunction for the quantum harmonic oscillator and let us determine the conditions of L-order classicality for $|\phi\rangle$. The first step is to determine the fundamental sequences (1). They are the single-value sequences $S_1 = q$ and $S_2 = p$. For the L-order classical conditions the relevant sequences are arrays of L fundamental sequences:

$$S^L = (z_1, \dots, z_L) \qquad z_i = q \vee p \qquad i = 1 \dots L \tag{5}$$

and the condition of L-order classicality (3) reads $\langle E_{z_1,\dots,z_L} | E_{z_1,\dots,z_L}\rangle \leqslant \delta^2_{z_1}(0) \dots \delta^2_{z_L}(0)$, $\forall S^L = (z_1, \dots, z_L)$ in (5), which is equivalent to

$$\langle\phi|(\hat{z}_1 - z_1(0)) \dots (\hat{z}_L - z_L(0))(\hat{z}_L - z_L(0)) \dots (\hat{z}_1 - z_1(0))|\phi\rangle \leqslant \delta^2_{z_1}(0) \dots \delta^2_{z_L}(0). \tag{6}$$

Using the Shwartz inequality and disregarding the contribution of terms proportional to $\hbar^2$, the former inequalities are reduced to

$$\begin{cases} \langle\phi|(\hat{q} - q(0))^{2L}|\phi\rangle \leqslant \delta_q(0)^{2L} \\ \langle\phi|(\hat{p} - p(0))^{2L}|\phi\rangle \leqslant \delta_p(0)^{2L} \end{cases} \Longleftrightarrow \begin{cases} \displaystyle\int (q - q(0))^{2L} |\phi(q)|^2 \, \mathrm{d}q \leqslant \delta_q(0)^{2L} \\ \displaystyle\int (p - p(0))^{2L} |\phi(p)|^2 \, \mathrm{d}p \leqslant \delta_p(0)^{2L}. \end{cases} \tag{7}$$

Given the classical initial data $\{q(0),\ p(0),\ \delta_q(0),\ \delta_p(0)\}$, (7) constitutes a system of inequalities to be satisfied by initial data wavefunction $|\phi\rangle$. Its set of solutions is $\{|\phi\rangle\}_L$. The higher the order of classicality L the more restrictive is the former system. For typical values of $\delta_q(0), \delta_p(0)$ (and choosing L of reasonable size) there are many solutions of (7). Gaussian wavepackets, for instance, provide well known solutions:

$$\phi_{\mathrm{G}}(q_0,\ p_0,\ \Delta q,\ q) = \frac{1}{(2\Pi(\Delta q)^2)^{1/4}} \exp\left\{-\frac{(q-q_0)^2}{4(\Delta q)^2} + \mathrm{i}p_0 q/\hbar\right\}. \tag{8}$$

In fact, if we take the parameters q_0 and p_0 to be given by $q_0 = q(0)$ and $p_0 = p(0)$, and substitute ϕ by $\phi_{\mathrm{G}}(q(0),\ p(0),\ \Delta q,\ q)$ in (7) we obtain

$$\frac{(2L-1)!}{2^{L-1}((L-1)!)}(\Delta q)^{2L} \leqslant \delta_q(0)^{2L} \wedge \frac{(2L-1)!}{2^{L-1}((L-1)!)}\left(\frac{\hbar}{2^{1/2}\Delta q}\right)^{2L} \leqslant \delta_p(0)^{2L}. \tag{9}$$

We notice that the higher the value of L the more confined will be the wavefunction $|\phi\rangle$ (in particular the Gaussian wavefunction $|\phi_{\mathrm{G}}\rangle$) in the representation of both $\hat{q}$ and $\hat{p}$.

The main result of [15] is that all wavefunctions $|\phi\rangle \in \{|\phi\rangle\}_L$ display a time evolution satisfying the following property for all times:

$$\int_{q(t)-\frac{\delta_q(t)}{(1-P)^{1/2L}}}^{q(t)+\frac{\delta_q(t)}{(1-P)^{1/2L}}} |\phi(q,t)|^2\,\mathrm{d}q \geqslant P \wedge \int_{p(t)-\frac{\delta_p(t)}{(1-P)^{1/2L}}}^{p(t)+\frac{\delta_p(t)}{(1-P)^{1/2L}}} |\phi(p,t)|^2\,\mathrm{d}p \geqslant P \qquad \forall_{0\leqslant P<1} \tag{10}$$

where $q(t),\ p(t),\ \delta_q(t),\ \delta_p(t)$ are given by (4), $|\phi(t)\rangle$ is the solution of the Schrödinger equation

$$\mathrm{i}\hbar\partial/\partial t|\phi(t)\rangle = 1/2(\hat{q}^2+\hat{p}^2)|\phi(t)\rangle \qquad |\phi(t=0)\rangle = |\phi\rangle \tag{11}$$

and P is an arbitrary probability. Take for instance $L = 1$ and $P = 0.99$, equation (10) states that 99% of the probability of the wavefunction $|\phi(t)\rangle$, in the representations of both $\hat{q}$ and $\hat{p}$, is confined to the classical intervals $[q(t) - 10\delta_q(t), q(t) + 10\delta_q(t)]$ and $[p(t) - 10\delta_p(t),\ p(t) + 10\delta_p(t)]$, respectively. This statement is valid for all times and, for the case of Gaussian wavepackets it can be easily verified by numerical computation of the integrals (10).

To see what happens when we increase the degree of classicality let us now consider $L = 10$. The classicality conditions are given by (7) with $L = 10$ (in particular, Gaussian solutions should satisfy (9) with $L = 10$). The time evolution of an arbitrary 10th-order classical wavefunction satisfies (10) with $L = 10$. In particular for $P = 0.99$ we have $(1-P)^{1/20} = 0.79$ and thus

$$\int_{q(t)-1.25\delta_q(t)}^{q(t)+1.25\delta_q(t)} |\phi(q,t)|^2\,\mathrm{d}q \geqslant 0.99 \wedge \int_{p(t)-1.25\delta_p(t)}^{p(t)+1.25\delta_p(t)} |\phi(p,t)|^2\,\mathrm{d}p \geqslant 0.99 \tag{12}$$

where $|\phi(t)\rangle$ is given by (11) and the result is valid for all times. This concludes the example.

For an arbitrary dynamical system the classicality conditions determine a class of wavefunctions that display a time evolution consistent with the classical predictions. Such conditions seem thus suitable to be used here to determine the class of wavefunctions $|\phi^{\mathrm{c}}\rangle$ compatible with the classical sector initial data.

In conclusion, the true physical configuration of the classical sector is given by a wavefunction $|\phi^{\mathrm{c}}\rangle$. However, we are only given the classical imprecise description (O_i^0, δ_i) and thus we should not assume that we know $|\phi^{\mathrm{c}}\rangle$ completely. If we assume that the classical initial data is L-order valid then any wavefunction belonging to the class of L-order classical wavefunctions can be, up to what we know, the true physical configuration of the system. Therefore, the classical sector initial configuration is properly described, not by a single wavefunction, but by the class of L-order classical wavefunctions $\{|\phi^{\mathrm{c}}\rangle\}_L$.

The initial-time configuration of the other sector of the half quantum system is described by standard quantum data. That is, we supply a completely fixed initial-time wavefunction:

$$|\phi^{\mathrm{Q}}\rangle = \sum_{k_1,\ldots,k_N} C_{k_1,\ldots,k_N}|k_1,\ldots,k_N\rangle. \tag{13}$$

The final step is to put the two sectors together and obtain the total wavefunction. To make it simple we assume that there is no kinematical coupling between the two sectors and thus the initial data wavefunction is of the form

$$|\phi\rangle = |\phi^{\mathrm{Q}}\rangle|\phi^{\mathrm{c}}\rangle = \sum_{k_1,\ldots,k_N} C_{k_1,\ldots,k_N}|k_1,\ldots,k_N\rangle|\phi^{\mathrm{c}}\rangle \tag{14}$$

where $|\phi^{\mathrm{c}}\rangle \in \{|\phi^{\mathrm{c}}\rangle\}_L$ is an L-order classical wavefunction. The set of all initial data wavefunctions of the form (14) will be designated by $\{|\phi\rangle\}_L$. That is, $\{|\phi\rangle\}_L = \{|\phi\rangle = |\phi^{\mathrm{Q}}\rangle|\phi^{\mathrm{c}}\rangle \in \mathcal{H} : |\phi^{\mathrm{c}}\rangle \in \{|\phi^{\mathrm{c}}\rangle\}_L\}$.

2.2. From quantum mechanics to half quantum mechanics—dynamics

The aim now is to obtain the time evolution of the initial data wavefunction (14). To do this let us work in the Heisenberg picture and calculate the full quantum time evolution of an arbitrary fundamental observable $\hat{O}_k$:

$$\hat{O}_k(t) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{t}{\mathrm{i}\hbar}\right)^n [\ldots[\hat{O}_k, \hat{H}], \ldots, \hat{H}]. \tag{15}$$

Let us designate the general operator $\hat{O}_k(t_0)$ just by $\hat{A}$. The aim is then to study the functional form of the initial data wavefunction in the representation of $\hat{A}$. The first step is to write the general observable $\hat{A}$ as a sum of multiple products of the fundamental observables:

$$\hat{A} = \sum_j \hat{A}^{\mathrm{c}}_j \hat{A}^{\mathrm{Q}}_j : \begin{cases} \hat{A}^{\mathrm{c}}_j = c_j \prod_{a=1}^{n_j} \hat{O}^{\mathrm{c}}_{i_a(j)} \\ \hat{A}^{\mathrm{Q}}_j = \prod_{b=1}^{m_j} \hat{O}^{\mathrm{Q}}_{\alpha_b(j)} \end{cases} \tag{16}$$

where for each j the sets of coefficients $i_a(j)$ and $\alpha_b(j)$ are two sequences, the first one having values in $\{1\ldots 2M\}$ and the second one in $\{2M+1\ldots 2(M+N)\}$, being c_j complex parameters that may depend on time. Let us proceed naively and try to obtain predictions for the outputs of a measurement of $\hat{A}$. Let then $|a^n_i\rangle$ be the general eigenvector of $\hat{A}$ with associated eigenvalue a_i and degeneracy index n. Using the standard prescription the predictions are given by the set of pairs $(a_i, P(a_i))$ where $P(a_i) = \sum_n |\langle a^n_i|\phi\rangle|^2$. We easily realize that we have a problem. In fact we do not know $|\phi\rangle$ completely and so the calculation of $P(a_i)$ is, to say the least, not straightforward.

To circumvent the problem we introduce a new operator $\hat{B}$ obtained by applying a map V_0, named unquantization, to the operator $\hat{A}$. This map V_0 is defined as a trivial extension of the full unquantization map (mapping quantum operators to full classical observables) that was defined and studied in a related paper. Let us then present the definition of V_0:

Definition 1 (First unquantization map). *Let $\mathcal{A}(\mathcal{H})$ be the algebra of linear operators acting on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and let $\mathcal{S}$ be the algebra of C^∞ functionals $\mathcal{S} = \{f : T^*M_1 \longrightarrow \mathcal{A}(\mathcal{H}_2)\}$. The unquantization V_0 is a map from $\mathcal{A}(\mathcal{H})$ to $\mathcal{S}$ that satisfies the following rules (we use the notation of (16)):*

(1) $V_0(\sum_j \hat{A}^{\mathrm{c}}_j \hat{A}^{\mathrm{Q}}_j) = \sum_j V_0(\hat{A}^{\mathrm{c}}_j) V_0(\hat{A}^{\mathrm{Q}}_j)$.

(2) $V_0(\hat{A}_j^{\mathrm{Q}}) = \hat{A}_j^{\mathrm{Q}}$.

(3) $V_0(\hat{A}_j^{\mathrm{c}}) = A_j^{\mathrm{c}}$. *The unquantization map that take us from $\hat{A}_i^{\mathrm{c}}$ to A_i^{c} was defined in [15], when the intention was to derive the full classical observable that corresponds to a general quantum operator. The following steps defined this procedure: (i) $\hat{A}_j^{\mathrm{c}}$ should be expanded as a sum of a Hermitian operator and an anti-Hermitian one, (ii) all antisymmetric terms of $\hat{A}_j^{\mathrm{c}}$ should then be executed, i.e. all the commutators present in $\hat{A}_j^{\mathrm{c}}$ should be calculated, and (iii) finally, given $\hat{A}_j^{\mathrm{c}}$ displayed in an order satisfying the two previous requirements, we perform the substitution of the quantum fundamental operators present in $\hat{A}_j^{\mathrm{c}}$ by the corresponding classical canonical variables, i.e. if $\hat{A}_j^{\mathrm{c}} = F(\hat{O}_i)$ where F satisfies the order requirements (i) and (ii) then $A_j^{\mathrm{c}} = F(O_i)$.*

Notice that the map V_0 is beset by order problems. In section 3 we will study the properties of V_0 in detail and in particular, we will see that these order problems do not affect the results of this section. Moreover we will also see that V_0 is just a trivial extension of the Dirac unquantization map, i.e. for an arbitrary classical sector observable A^{c} we have $V_0 \circ \wedge(A^{\mathrm{c}}) = A^{\mathrm{c}}$ where $\wedge$ is the standard Dirac quantization map [16, 17]. To proceed we apply V_0 to $\hat{A}$ and obtain

$$\hat{B} = \sum_j A_j^{\mathrm{c}} \hat{A}_j^{\mathrm{Q}}. \tag{17}$$

Let us elaborate on the reasons why we are introducing this new operator. The answer is twofold: firstly, $\hat{B}$ is the most similar operator to $\hat{A}$ for which we are able to calculate probabilities once the half quantum initial data are given. In fact to calculate $P(b_u) = \sum_r |\langle b_u^r|\phi^{\mathrm{Q}}\rangle|^2$—where $|b_u^r\rangle$ is a complete set of eigenstates of $\hat{B}(a_i^0)$ (spanning the Hilbert space $\mathcal{H}_2$) with associated eigenvalue b_u and degeneracy index r—we only need to know the half quantum initial data $(a_i^0, |\phi^{\mathrm{Q}}\rangle)$. Secondly, if $|\phi^{\mathrm{c}}\rangle$ is L-order classical with respect to the classical data (a_i^0, δ_i) then the probabilities $P(b_u)$ associated with $\hat{B}(a_i^0)$ and $|\phi^{\mathrm{Q}}\rangle$ can be used to obtain an upper and lower limit for the probabilities $P(a_i)$ associated with $\hat{A}$ and $|\phi^{\mathrm{Q}}\rangle|\phi^{\mathrm{c}}\rangle$.

Let us make this second statement more precise: let us consider an interval $I = [a^0 - D, a^0 + D]$ of eigenvalues of $\hat{A}$. The aim is then to calculate the probabilities $P(a_i \in I) = \sum_{a_i \in I} \sum_n |\langle a_i^n|\phi\rangle|^2$, which are the physical predictions of the full quantum treatment of the system. Since we do not have any information about the functional form of the wavefunction $|\phi^{\mathrm{c}}\rangle$ at scales smaller than the classical error margins it seems reasonable to only consider intervals with range D bigger than the classical error margins. Moreover, since $|\phi\rangle$ is not completely known (we only know that $|\phi\rangle \in \{|\phi\rangle\}_L$ (14)) our best chance is to calculate some upper and lower limits for the values of $P(a_i \in I)$—that is to calculate $P_{\max} \geqslant \max\{\sum_{n,a_i \in I} |\langle a_i^n|\phi\rangle|^2, |\phi\rangle \in \{|\phi\rangle\}_L\}$ and $P_{\min} \leqslant \min\{\sum_{n,a_i \in I} |\langle a_i^n|\phi\rangle|^2, |\phi\rangle \in \{|\phi\rangle\}_L\}$—and thus to obtain the predictions $P_{\min} \leqslant P(a_i \in I) \leqslant P_{\max}$. The main point of this section will be to prove that $P_{\max}$ and $P_{\min}$ can be obtained from the probabilities $P(b_u) = \sum_r |\langle b_u^r|\phi^{\mathrm{Q}}\rangle|^2$, that is from the half operator $\hat{B}$ and the half quantum initial data $(a_i^0, |\phi^{\mathrm{Q}}\rangle)$.

This is the desired result. The consequences are that firstly we are able to make predictions for the outputs of a measurement of the physical observable $\hat{A}$ when the initial state of the system is described not by a single wavefunction but by a class of wavefunctions $\{|\phi\rangle\}_L$, and secondly, and most important, these predictions are obtained using only half quantum objects and thus the half quantum treatment of the system is given a precise physical meaning in terms of the probabilities for the outputs of a measurement of the *real* physical observable $\hat{A}$.

To obtain $P(a_i \in I)$ from $\hat{B}$ and the half quantum initial data we will follow a three-step procedure:

(1) in the first step (section 2.2.1) the relation between $\hat{A}$ and $\hat{B}$ is made more explicit. More precisely, the aim is to expand $\hat{A} - \hat{B}$ in powers of the monomials $\hat{O}_i - O_i$.

(2) In section 2.2.2 the former expansion is used to study the functional form of the eigenstates $|\psi_u^r\rangle = |b_u^r\rangle|\phi^{\mathrm{c}}\rangle$ of $\hat{B}$ in the representation of $\hat{A}$ (in fact, for technical reasons we will introduce a slightly more general set of states than $|\psi_u^r\rangle$, but this is of no relevance now). We will calculate the maximum value of the spread of a general state of the form $|\psi_u^r\rangle = |b_u^r\rangle|\phi^{\mathrm{c}}\rangle$, $|\phi^{\mathrm{c}}\rangle \in \{|\phi^{\mathrm{c}}\rangle\}_L$ explicitly as a function of the half quantum operator $\hat{B}$ and the half quantum initial data $(|\phi^{\mathrm{Q}}\rangle, a_i^0, \delta_i)$. The former result is independent of the detailed functional form of $|\phi^{\mathrm{c}}\rangle$ (providing the wavefunction satisfies the L-order classicality conditions) and further it will show that the states $|\psi_u^r\rangle$ are very peaked in the representation of $\hat{A}$ (the more peaked, the higher the order of classicality of $|\phi^{\mathrm{c}}\rangle$ and the smaller initial data classical error margins).

(3) Roughly, what the former property tells us is that $|\psi_u^r\rangle$ is 'almost' an eigenstate of $\hat{A}$, i.e. '$|\psi_u^r\rangle \simeq |a_i^n\rangle$', and thus the values of $\sum_r |\langle\psi_u^r|\phi\rangle|^2$ and $\sum_n |\langle a_i^n|\phi\rangle|^2$ should be somehow related. Clearly, these are very inaccurate statements and making them precise requires some quite long calculations. In section 2.2.3 we will prove that: $P(b_u \in I) - E \leqslant P(a_i \in I) \leqslant P(b_u \in I) + E$, where $P(b_u) = \sum_r |\langle\psi_u^r|\phi\rangle|^2 = \sum_r |\langle b_u^r|\phi^{\mathrm{Q}}\rangle|^2$ and the value of the error E is proportional to the spread of the states $|\psi_u^r\rangle$ in the representation of $\hat{A}$. Both $P(b_u \in I)$ and E are half quantum functions and thus we succeed in obtaining the values of $P_{\max} = P(b_u \in I) + E$ and $P_{\min} = P(b_u \in I) - E$ from the half quantum objects.

2.2.1. Relating $\hat{A}$ and $\hat{B}$. Let us consider the general classical sector operator $\hat{A}_j^{\mathrm{c}}$ and the corresponding classical observable A_j^{c}. Since $\hat{A}_j^{\mathrm{c}}$ and A_j^{c} have the same functional form the following relation is valid up to a correction term of order of $\hbar^2$:

$$\hat{A}_j^{\mathrm{c}} - A_j^{\mathrm{c}} = \sum_{i=1}^{2M} \frac{\partial A_j^{\mathrm{c}}}{\partial O_i}(\hat{O}_i - O_i) + \frac{1}{2}\sum_{i,k=1}^{2M} \frac{\partial^2 A_j^{\mathrm{c}}}{\partial O_i \partial O_k}(\hat{O}_i - O_i)(\hat{O}_k - O_k) + \cdots. \tag{18}$$

This expansion was derived and discussed in detail in [15]. There we point out that (18) is exactly valid only if A_j^{c} is obtained from a totally symmetric form of $\hat{A}_j^{\mathrm{c}}$. This is not the general case if we use the unquantization map V_0 to obtain A_j^{c}. In fact we saw in [15] that if we use the map V_0—in which case A_j^{c} is obtained from $\hat{A}_j^{\mathrm{c}}$ displayed in an order that does not contain antisymmetric components (see definition 1)—then the difference between the two sides of equation (18) is given by a leading term of the form $c_j\hbar^2\hat{\epsilon}_j$, where c_j is the numerical factor in $\hat{A}_j^{\mathrm{c}}$ (16) and $\hat{\epsilon}_j$ is the 'operator error' proportional to a sum of products of monomials $(\hat{O}_i - O_i)$, each of the products having at most $n_j - 2$ terms (check equation (16) for the meaning of n_j). We explicitly included the correction term $c_j\hbar^2\hat{\epsilon}_j$ in (18) and obtain

$$\begin{aligned}(\hat{A} - \hat{B})^L &= \left\{ \sum_j (\hat{A}_j^{\mathrm{c}} - A_j^{\mathrm{c}})\hat{A}_j^{\mathrm{Q}} \right\}^L \\ &= \left\{ \sum_{i=1}^{2M} \frac{\partial \hat{B}}{\partial O_i}(\hat{O}_i - O_i) + \frac{1}{2}\sum_{i,k=1}^{2M} \frac{\partial^2 \hat{B}}{\partial O_i \partial O_k}(\hat{O}_i - O_i)(\hat{O}_k - O_k) \right. \\ &\quad \left. + \cdots + \sum_j c_j\hbar^2\hat{\epsilon}_j\hat{A}_j^{\mathrm{Q}} \right\}^L. \end{aligned} \tag{19}$$

Finally, to lowest order this expansion can be cast in the form

$$(\hat{A}-\hat{B})^L=\sum_{i_1=1}^{2M}\cdots\sum_{i_L=1}^{2M}\prod_{s=1}^{L}\frac{\partial\hat{B}}{\partial O_{i_s}}(\hat{O}_{i_s}-O_{i_s})+\cdots \tag{20}$$

where we disregard the contribution of the term proportional to $c_j\hbar^2$. Typically, this contribution is meaningless when compared to the terms proportional to the derivatives of $\hat{B}$. However in some artificial examples this may not be the case. Consider for instance $\hat{A}=1/2(\hat{x}^{\mathrm{c}}\hat{y}^{\mathrm{c}}\hat{z}^{\mathrm{c}}+\hat{z}^{\mathrm{c}}\hat{y}^{\mathrm{c}}\hat{x}^{\mathrm{c}})\hat{A}^{\mathrm{Q}}-1/2(\hat{y}^{\mathrm{c}}\hat{x}^{\mathrm{c}}\hat{z}^{\mathrm{c}}+\hat{z}^{\mathrm{c}}\hat{x}^{\mathrm{c}}\hat{y}^{\mathrm{c}})\hat{A}^{\mathrm{Q}}$, where $\hat{x}^{\mathrm{c}}$, $\hat{y}^{\mathrm{c}}$, $\hat{z}^{\mathrm{c}}$ are Hermitian, classical sector operators of an arbitrary system. We have $V_0(\hat{A})=xyz\hat{A}^{\mathrm{Q}}-xyz\hat{A}^{\mathrm{Q}}=0=\hat{B}$ and therefore, in this case, $\hat{A}-\hat{B}=\sum_j c_j\hbar^2\hat{\epsilon}_j\hat{A}^{\mathrm{Q}}_j$ which, in general, is not zero. The problem lies, of course, in the order in which $\hat{A}$ is displayed before we apply the map V_0. One should impose the restriction that $\hat{A}$ cannot be displayed in an order in which some unresolved commutators are present. One easy way to check that this is the case is precisely by comparing the magnitude of $\hat{B}$ (the numerical factors in $\hat{B}$) with the magnitude of $c_j\hbar^2$, where the c_j are the numerical factors of $\hat{A}$. For physical relevant examples (physical Hamiltonians and observables), namely for the time evolution of a general quantum observable, it is easy to verify whether the original operator $\hat{A}$ is in an adequate order (this is in fact the typical case), and thus, upon unquatization, one has *magnitude* $(\hat{B})\propto c_j\hbar^0\gg c_j\hbar^2$. Therefore, and keeping the caution remark in mind, we shall take the result (20) to be exactly valid.

2.2.2. Relating the eigenstates of $\hat{B}$ to the eigenstates of $\hat{A}$. Let us start by introducing the states $|\psi^r_u\rangle=|\phi^{\mathrm{c}}\rangle|b^r_u\rangle$ where $|b^r_u\rangle$ form a complete set of eigenstates of $\hat{B}$ (spanning the Hilbert space $\mathcal{H}_2$) and $|\phi^{\mathrm{c}}\rangle\in\{|\phi^{\mathrm{c}}\rangle\}_L$ is the classical sector initial data wavefunction. For reasons that will became clear later on we will also introduce the more general set of states of the form

$$|\xi_u\rangle=|\xi^{\mathrm{Q}}_u\rangle|\phi^{\mathrm{c}}\rangle=\frac{1}{C_u}\sum_{r,b'_u\in I_u}\langle b'^r_u|\phi^{\mathrm{Q}}\rangle|b'^r_u\rangle|\phi^{\mathrm{c}}\rangle \tag{21}$$

where $C_u=\langle\phi|\xi_u\rangle$ is a normalization constant, $|b'^r_u\rangle$ are eigenstates of $\hat{B}$ and $I_u=[b_u-I_B,b_u+I_B]$ where b_u is named the *central eigenvalue* associated with $|\xi_u\rangle$ and I_B is a constant to be supplied later, that represents the spread of $|\xi_u\rangle$ in the representation of $\hat{B}$. We are specially interested in a set of states $|\xi_u\rangle$ associated with a sequence S of eigenvalues b_u of $\hat{B}$. These eigenvalues are chosen in such a way that their value grows in steps of $2I_B$. This way we guarantee that firstly $\langle\xi_u|\xi_{u'}\rangle=\delta_{u,u'}$ and secondly that $|\phi\rangle=\sum_{b_u\in S}\langle\xi_u|\phi\rangle|\xi_u\rangle$.

The set of states $|\xi_u\rangle$ is in some sense a generalization of the set of eigenstates $|\psi^r_u\rangle$. In fact, we notice that for $I_B=0$ the set $\{|\xi_u\rangle\}$ reduces to a set of true eigenstates of $\hat{B}$. Moreover they can be used to obtain the probabilities of $|\phi\rangle$ in the representation of $\hat{B}$, providing we only consider intervals of eigenvalues I of range multiple of I_B:

$$P(b'_u\in I)=\sum_{r,b'_u\in I}|\langle\phi|b'^r_u\rangle|\phi^{\mathrm{c}}\rangle|^2=\sum_{b_u\in I\cap S}|\langle\xi_u|\phi\rangle|^2. \tag{22}$$

The reason we are introducing these new states is that they will be used (in section 2.2.3) to obtain the probabilities of $\hat{A}$ as a function of the probabilities of $\hat{B}$. As we will see, such a relation cannot be obtained, because of larger interference effects, if we use the states $|\psi^r_u\rangle$.

This said, let us obtain the maximum value of the L-order spread of $|\xi_u\rangle$ in the representation of $\hat{A}$ (the reader should refer to the appendix for the relevant definitions).

Theorem. *If $|\phi^{\mathrm{c}}\rangle$ is an L-order classical wavefunction with respect to the classical data (a^0_i,δ_i) then the L-order spread $\Delta_L(\hat{A},\xi_u,b_u,p)$ of the general state $|\xi_u\rangle$ in the representation of $\hat{A}$*

satisfies

$$\Delta_L(\hat{A}, \xi_u, b_u, p) \leqslant \frac{\delta_L(\hat{B}) + I_B}{(1-p)^{1/2L}} \tag{23}$$

where $\delta_L(\hat{B})$*, named the* L*-order error margin of* $\hat{B}$*, is given by*

$$\begin{aligned}\delta_L(\hat{B}) = \sum_{i=1}^{2M} \left| \langle \xi_u^{\mathrm{Q}} | \left(\frac{\partial \hat{B}^\dagger}{\partial O_i} \right)^L \left(\frac{\partial \hat{B}}{\partial O_i} \right)^L | \xi_u^{\mathrm{Q}} \rangle \right|^{1/2L} \delta_i \\ + \cdots + \frac{1}{n!} \sum_{i,k,\ldots,s=1}^{2M} \left| \langle \xi^{\mathrm{Q}} | \left(\frac{\partial^n \hat{B}^\dagger}{\partial O_i \partial O_k \cdots \partial O_s} \right)^L \left(\frac{\partial^n \hat{B}}{\partial O_i \partial O_k \cdots \partial O_s} \right)^L | \xi^{\mathrm{Q}} \rangle \right|^{1/2L} \\ \times \delta_i \delta_k \ldots \delta_s \end{aligned} \tag{24}$$

and the reason $\delta_L(\hat{B})$ *was named the 'L-order error margin of* $\hat{B}$*' is the resemblance of its functional form to the standard error margin of an arbitrary classical observable:* $\delta_B = \sum_{i=1}^{2M} |\partial B/\partial O_i| \delta_i + \cdots + \frac{1}{n!} \sum_{i,k,\ldots,s=1}^{2M} |\partial^n B/\partial O_i \partial O_k \cdots \partial O_s| \delta_i \delta_k \cdots \delta_s$.

Proof. The L-order spread

$$\frac{\Delta_L(\hat{A}, \xi_u, b_u, p) = \langle E^L(\hat{A}, \xi_u, b_u) | E^L(\hat{A}, \xi_u, b_u) \rangle^{1/2L}}{(1-p)^{1/2L}}$$

can be cast in the form

$$\Delta_L(\hat{A}, \xi_u, b_u, p) = \frac{|\langle \xi_u | (\hat{A} - b_u)^{2L} | \xi_u \rangle|^{1/2L}}{(1-p)^{1/2L}} = \frac{|\langle \xi_u | \{(\hat{A} - \hat{B}) + (\hat{B} - b_u)\}^{2L} | \xi_u \rangle|^{1/2L}}{(1-p)^{1/2L}}. \tag{25}$$

Expanding the polynomial inside the bracket and using the Schwartz inequality L times to separate the terms in $(\hat{A} - \hat{B})$ of those in $(\hat{B} - b_u)$ we obtain

$$\Delta_L(\hat{A}, \xi_u, b_u, p) \leqslant \frac{1}{(1-p)^{1/2L}} \{ |\langle \xi_u | (\hat{A} - \hat{B})^{2L} | \xi_u \rangle|^{1/2L} + |\langle \xi_u | (\hat{B} - b_u)^{2L} | \xi_u \rangle|^{1/2L} \} \tag{26}$$

where we disregard the contribution of terms proportional to $\hbar^2$ or smaller. For the first average value in (26) we use the expansion (20) and easily find that

$$\begin{aligned}|\langle \xi_u | (\hat{A} - \hat{B})^{2L} | \xi_u \rangle| \leqslant \sum_{i_1=1}^{2M} \cdots \sum_{i_L=1}^{2M} \sum_{k_1=1}^{2M} \cdots \sum_{k_L=1}^{2M} |\langle E_{O_{k1},\ldots,O_{kL}} | E_{O_{i_1},\ldots,O_{iL}} \rangle| \\ \times \left| \langle \xi_u^{\mathrm{Q}} | \frac{\partial \hat{B}^\dagger}{\partial O_{k_1}} \cdots \frac{\partial \hat{B}^\dagger}{\partial O_{k_L}} \frac{\partial \hat{B}}{\partial O_{i_1}} \cdots \frac{\partial \hat{B}}{\partial O_{i_L}} | \xi_u^{\mathrm{Q}} \rangle \right| + \cdots . \end{aligned} \tag{27}$$

Using the Shwartz inequality L times, the relation (3) and disregarding the contributions of terms proportional to $\hbar^2$ or smaller we obtain $|\langle \xi_u | (\hat{A} - \hat{B})^{2L} | \xi_u \rangle|^{1/2L} \leqslant \delta_L(\hat{B})$. This result is valid up to any order since $|\phi^{\mathrm{c}}\rangle$ is L-order classical and so the relation $\langle E_{S_{ia}^L} | E_{S_{ia}^L} \rangle \leqslant (\delta_{S_{ia}^L})^2$ is valid for all the sequences $S_{i_a}^L$ determined in (1), which are exactly those involved in the expansion (27) of $|\langle \xi_u | (\hat{A} - \hat{B})^{2L} | \xi_u \rangle|$.

Replacing the former result in (26) and using, for the second average value of (26), the inequality $\langle \xi_u | (\hat{B} - b_u)^{2L} | \xi_u \rangle = \langle E^L(\hat{B}, \xi_u^{\mathrm{Q}}, b_u) | E^L(\hat{B}, \xi_u^{\mathrm{Q}}, b_u) \rangle \leqslant I_B^{2L}$, which can easily be obtained from (21), we finally obtain the relation (23), proving the theorem. □

Two remarks are in order. Firstly, notice that the value of the spread $\Delta_L(\hat{A}, \xi_u, b_u, p)$ is given as an exclusive function of half quantum quantities: we are not required to supply

the wavefunction $|\phi^c\rangle$ or the full quantum operator $\hat{A}$ and yet the spread is that of $|\xi_u\rangle$ *in the representation* of $\hat{A}$. Secondly, we point out that using the result (a) of the appendix we can state that in the representation of $\hat{A}$ the state $|\xi_u\rangle$ has at least a probability p in the interval $I = [b_u - \Delta_L(p), b_u + \Delta_L(p)]$, with $\Delta_L(p)$ given by (23), that is $\sum_{n,a_i^n \in I} |\langle a_i^n|\xi_u\rangle|^2 \geqslant p$. Take for instance the classical initial data to be 10th-order valid, $L = 10$, and let us choose $p = 0.99$. We have $1/(1-p)^{1/2L} = 1.25$ and thus the statement is that 99% of the probability of $|\xi_u\rangle$ in the representation of $\hat{A}$ is confined to the interval $[b_u - 1.25(\delta_{10}(\hat{B}) + I_B), b_u + 1.25(\delta_{10}(\hat{B}) + I_B)]$, where b_u is the central eigenvalue associated with $|\xi_u\rangle$ and $\delta_{10}(\hat{B})$ is given by (24). For small values of the classical error margins $|\xi_u\rangle$ are very peaked states in the representation of $\hat{A}$ and we suspect that we might be able to use them, instead of the true eigenvectors of $\hat{A}$, to obtain the physical probabilities $P(a_i \in I)$. This will be the aim of the next section.

2.2.3. Relating the probabilities in the representation of $\hat{A}$ with the probabilities in the representation of $\hat{B}$. Finally, the aim is to obtain predictions for the probability of a measurement of $\hat{A}$ yielding an eigenvalue $a_i \in I_0$, where I_0 is an interval of size at least Δ_L, with Δ_L given by (23). More precisely our predictions will be that the probability $P(a_i \in I_0)$ is at the most $P_{\max}(a_i \in I_0)$ and at the least $P_{\min}(a_i \in I_0)$, the error margin being a function of L, typically of reasonable size.

Let us then consider the following three intervals. $I_0 = [a^0 - D, a^0 + D]$ is the interval of eigenvalues of $\hat{A}$ for which we want to determine $P(a_i \in I_0)$. D is required to satisfy $D > \Delta_L$. The two other intervals $I_{\max}$ and $I_{\min}$ will be needed to majorate and minorate the former probability. $I_{\max}$ is such that any $|\xi_u\rangle$ with associated central eigenvalue $b_u \in I_0$ has in the representation of $\hat{A}$ a probability of at least p in $I_{\max}$. For $I_{\min}$ the statement is that if $b_u \in I_{\min}$ then $|\xi_u\rangle$ has a probability of at least p in I_0. We easily see that the intervals $I_{\min} = [a^0 - (D - \Delta_L), a^0 + (D - \Delta_L)]$ and $I_{\max} = [a^0 - (D + \Delta_L), a^0 + (D + \Delta_L)]$—where $\Delta_L = \Delta_L(\hat{A}, \xi_u, b_u, p)$ is given by (23) and $b_u \in I_0$—will satisfy the former requirements. Notice that we have assumed that Δ_L has a similar value for different b_u within I_0. If this is not the case all the future results are still valid; we just need to be more careful in constructing the intervals $I_{\max}$ and $I_{\min}$. This said, let us then calculate the probabilities $P(a_i \in I)$. We have

$$P(a_i \in I_0) = \sum_{n,a_i \in I_0} |\langle \phi | a_i^n \rangle|^2 = \sum_{n,a_i \in I_0} \left| \sum_{b_u \in S} \langle \phi | \xi_u \rangle \langle \xi_u | a_i^n \rangle \right|^2 \tag{28}$$

where we have used the fact that $|\phi\rangle = \sum_{b_u \in S} \langle \xi_u | \phi \rangle |\xi_u\rangle$. We now expand the previous expression first using the interval $I_{\max}$:

$$\begin{aligned} P(a_i \in I_0) &= \sum_{n,a_i \in I_0} \left| \sum_{b_u \in I_{\max} \cap S} \langle \phi | \xi_u \rangle \langle \xi_u | a_i^n \rangle + \sum_{b_u \in S/I_{\max}} \langle \phi | \xi_u \rangle \langle \xi_u | a_i^n \rangle \right|^2 \\ &\leqslant \sum_{n,a_i} \left| \sum_{b_u \in I_{\max} \cap S} \langle \phi | \xi_u \rangle \langle \xi_u | a_i^n \rangle \right|^2 + \sum_{n,a_i \in I_0} \left| \sum_{b_u \in S/I_{\max}} \langle \phi | \xi_u \rangle \langle \xi_u | a_i^n \rangle \right|^2 \\ &\quad + 2 \left| \sum_{n,a_i \in I_0} \sum_{b_u \in I_{\max} \cap S} \langle \phi | \xi_u \rangle \langle \xi_u | a_i^n \rangle \sum_{b_u \in S/I_{\max}} \langle a_i^n | \xi_u \rangle \langle \xi_u | \phi \rangle \right| \end{aligned} \tag{29}$$

where the set $S/I_{\max}$ is constituted by the elements of S that do not belong to $I_{\max}$. On the other hand, and using the interval $I_{\min}$, we have $P(a_i \in I_0) = 1 - P(a_i \notin I_0)$ and so

$$P(a_i \in I_0) = 1 - \sum_{n,a_i \notin I_0} \left| \sum_{b_u \in S/I_{\min}} \langle \phi | \xi_u \rangle \langle \xi_u | a_i^n \rangle + \sum_{b_u \in I_{\min} \cap S} \langle \phi | \xi_u \rangle \langle \xi_u | a_i^n \rangle \right|^2$$

$$\geqslant 1-\sum_{n,a_i}\left|\sum_{b_u\in S/I_{\min}}\langle\phi|\xi_u\rangle\langle\xi_u|a_i^n\rangle\right|^2-\sum_{n,a_i\notin I_0}\left|\sum_{b_u\in I_{\min}\cap S}\langle\phi|\xi_u\rangle\langle\xi_u|a_i^n\rangle\right|^2$$
$$-2\left|\sum_{n,a_i\notin I_0}\sum_{b_u\in S/I_{\min}}\langle\phi|\xi_u\rangle\langle\xi_u|a_i^n\rangle\sum_{b_u\in I_{\min}\cap S}\langle a_i^n|\xi_u\rangle\langle\xi_u|\phi\rangle\right|. \tag{30}$$

The two former expansions provide the values of $P_{\max}$ and $P_{\min}$, respectively. They can be considerably simplified. Let us concentrate on the expansion (29). Using the fact that $\sum_{n,a_i}\langle\xi_u|a_i^n\rangle\langle a_i^n|\xi_v\rangle=\delta_{uv}$ and $\sum_{b_u\in I_{\max}\cap S}|\langle\phi|\xi_u\rangle|^2=\sum_{r,b_u\in I_{\max}}|\langle\phi|\psi_u^r\rangle|^2$, where $|\psi_u^r\rangle$ are the eigenstates of $\hat{B}$, we easily obtain

$$\sum_{n,a_i}\left|\sum_{b_u\in I_{\max}\cap S}\langle\phi|\xi_u\rangle\langle\xi_u|a_i^n\rangle\right|^2=\sum_{n,a_i}\sum_{b_u,b_v\in I_{\max}\cap S}\langle\phi|\xi_u\rangle\langle\xi_v|\phi\rangle\langle\xi_u|a_i^n\rangle\langle a_i^n|\xi_v\rangle$$
$$=\sum_{b_u\in I_{\max}\cap S}|\langle\phi|\xi_u\rangle|^2=P(b_u\in I_{\max}). \tag{31}$$

Taking the former result into account and using the Shwartz inequality we also obtain

$$\left|\sum_{n,a_i\in I_0}\sum_{b_u\in I_{\max}\cap S}\langle\phi|\xi_u\rangle\langle\xi_u|a_i^n\rangle\sum_{b_u\in S/I_{\max}}\langle a_i^n|\xi_u\rangle\langle\xi_u|\phi\rangle\right|$$
$$\leqslant P(b_u\in I_{\max})^{1/2}\left(\sum_{n,a_i\in I_0}\left|\sum_{b_u\in S/I_{\max}}\langle\phi|\xi_u\rangle\langle\xi_u|a_i^n\rangle\right|^2\right)^{1/2}. \tag{32}$$

Thus the expansion (29) can be cast in the form

$$P(a_i\in I_0)\leqslant\{P(b_u\in I_{\max})^{1/2}+X_1^{1/2}\}^2 \tag{33}$$

where $X_1=\sum_{n,a_i\in I_0}|\sum_{b_u\in S/I_{\max}}\langle\phi|\xi_u\rangle\langle\xi_u|a_i^n\rangle|^2$. For the expansion (30) a similar calculation yields

$$P(a_i\in I_0)\geqslant 1-\{P(b_u\notin I_{\min})^{1/2}+X_2^{1/2}\}^2 \tag{34}$$

where $X_2=\sum_{n,a_i\notin I_0}|\sum_{b_u\in I_{\min}\cap S}\langle\phi|\xi_u\rangle\langle\xi_u|a_i^n\rangle|^2$. We have found the first relation between the probabilities in the representation of $\hat{A}$ and the probabilities in the representation of $\hat{B}$.

To proceed we still have to calculate the maximum values of X_1 and X_2 and express these results in terms of half quantum quantities. The following theorem will do this:

Theorem. *If $|\phi^{\mathrm{c}}\rangle$ is an L-order classical initial data wavefunction then X_1 satisfies*

$$X_1\leqslant(1-p)\left\{\sum_{b_u\in S/I_{\max}}\frac{\Delta_L(\hat{A},\xi_u,b_u,p)^L|\langle\phi|\xi_u\rangle|}{|b_u-a|^L}\right\}^2 \tag{35}$$

where $\Delta_L(\hat{A},\xi_u,b_u,p)$ is given by (23) and a is one of the extremes of the interval I_0, the one that minimizes the distance $|b_u-a|$, that is $a=a^0+D$ or $a=a^0-D$. Moreover, the result is valid for all probabilities $0\leqslant p<1$ and for all sequences S of central eigenvalues of $\hat{B}$ (in particular S can be the set of true eigenvalues of $\hat{B}$).

If $\Delta_L(\hat{A},\xi_u,b_u,p)$ is approximately a constant in the range of eigenvalues where $|\langle\phi|\xi_u\rangle|$ have meaningful values the former result reduces to

$$X_1\leqslant(1-p)\frac{\Delta_L(p)}{2(2L-1)I_B} \tag{36}$$

where $\Delta_L(p)=\Delta_L(\hat{A},\xi_u,b_u,p)$ for any b_u for which $|\langle\phi|\xi_u\rangle|\not\approx 0$. Under the same conditions X_2 satisfies exactly the same relation: $X_2\leqslant(1-p)\frac{\Delta_L(p)}{2(2L-1)I_B}$.

Proof. We will concentrate on the case of X_1. A similar calculation can be performed for X_2. From the definition of X_1 we easily obtain

$$X_1 \leqslant \sum_{b_u, b_v \in S/I_{\max}} |\langle\phi|\xi_u\rangle\langle\xi_v|\phi\rangle| \left| \sum_{n, a_i^n \in I_0} \langle\xi_u|a_i^n\rangle\langle a_i^n|\xi_v\rangle \right|. \tag{37}$$

For the second term of the right-hand side of the previous inequality, we have

$$\left| \sum_{n, a_i^n \in I_0} \langle\xi_u|a_i^n\rangle\langle a_i^n|\xi_v\rangle \right| \leqslant \left(\sum_{n, a_i^n \in I_0} |\langle\xi_u|a_i^n\rangle|^2 \right)^{1/2} \left(\sum_{n, a_i^n \in I_0} |\langle\xi_v|a_i^n\rangle|^2 \right)^{1/2}. \tag{38}$$

Furthermore, we notice that both $b_u, b_v \notin I_{\max}$ and use the result (65) in (b) from the appendix to obtain

$$\sum_{n, a_i^n \in I_0} |\langle\xi_u|a_i^n\rangle|^2 \leqslant (1-p) \frac{\Delta_L(\hat{A}, \xi_u, b_u, p)^{2L}}{|b_u - a|^{2L}} \tag{39}$$

where a is one of the extremes of the interval I_0. Putting these results together it is straightforward to obtain (35), proving the first result of the theorem.

To proceed we consider the case in which $\Delta_L(\hat{A}, \xi_u, b_u, p)$ is approximately a constant in the range of eigenvalues where $|\langle\phi|\xi_u\rangle|$ have meaningful values. In this case (35) reduces to

$$X_1 \leqslant (1-p)\Delta_L(p)^{2L} \left\{ \sum_{b_u \in S/I_{\max}} \frac{|\langle\phi|\xi_u\rangle|}{|b_u - a|^L} \right\}^2. \tag{40}$$

The aim is to maximize the term inside the brackets to obtain the highest possible value of X_1. Let then $|\phi\rangle = \sum_{b_u \in S} \langle\xi_u|\phi\rangle|\xi_u\rangle$ and let $C_u = |\langle\xi_u|\phi\rangle|$. As an intermediate step let us assume that $|\phi\rangle$ spreads for an interval from $E_0 = a^0 + \Delta_L(p) + D$ (the extreme of $I_{\max}$) to an arbitrary $E \in \mathcal{R}$, i.e. $C_u = 0$ for $b_u \notin [E_0, E]$. In the end we will see that the result for X_1 is independent of E, that might be taken to infinity. Let us proceed: since b_u grows in steps of size $2I_B$, we divide the former interval into sub-intervals of size $2I_B$. Let us say that we have N such sub-intervals: $2I_B = (E - E_0)/N$. We then obtain

$$\begin{aligned} \sum_{b_u \in S/I_{\max}} \frac{|\langle\phi|\xi_u\rangle|}{|b_u - a|^L} &= \sum_{n=0}^{N} \frac{C_n}{(\Delta_L + 2nI_B)^L} = \frac{1}{2I_B} \sum_{n=0}^{N} 2I_B \frac{C_n}{(\Delta_L + 2nI_B)^L} \\ &\simeq \frac{1}{2I_B} \int_{E_0}^{E} \frac{C(x)}{|x - a|^L} \, \mathrm{d}x \end{aligned} \tag{41}$$

where $C_n = |\langle\phi|\xi_n\rangle|$ and $|\xi_n\rangle$ is the state associated to the central eigenvalue $b_n = a + \Delta_L(p) + 2nI_B$. Our task now is to maximize the previous integral subject to the constraint

$$\sum_{n=0}^{N} C_n^2 = 1 \Longrightarrow \frac{1}{2I_B} \int_{E_0}^{E} C(x)^2 \, \mathrm{d}x = 1. \tag{42}$$

This is easily done using the Lagrangian multiplier method. We obtain

$$\sum_{b_u \in S/I_{\max}} \frac{|\langle\phi|\xi_u\rangle|}{|b_u - a|^L} \leqslant \left(\frac{1}{2(2L-1)I_B(\Delta_L(p))^{2L-1}} \right)^{1/2}. \tag{43}$$

Substituting this result in (40) we finally obtain (36), proving the theorem. □

Two brief remarks are in order: firstly, to say that since $|\langle\phi|\xi_u\rangle| = |\langle\phi^{\mathrm{Q}}|\xi_u^{\mathrm{Q}}\rangle|$ the calculation of the right-hand side of inequality (35) might be done explicitly once the initial data of the half quantum system and the operator $\hat{B}$ are given and thus we may use this result to obtain the value of $P_{\max}$ from the exclusive knowledge of the half quantum quantities; secondly, to

point out that the result (36) could not have been obtained if we had used the eigenstates $|\psi_u^r\rangle$ of $\hat{B}$, in which case $I_B = 0$ and thus X_1 would not be bounded.

We now replace the maximum values of X_1 and X_2 in (33) and (34) and finally obtain

$$P(b_u \in I_{\min}) - E_{\min} \leqslant P(a_i \in I_0) \leqslant P(b_u \in I_{\max}) + E_{\max} \tag{44}$$

where $P(b_u \in I_{\min}) = \sum_{r,b_u \in I_{\min}} |\langle b_u^r|\phi^{\mathrm{Q}}\rangle|^2$, $P(b_u \in I_{\max}) = \sum_{r,b_u \in I_{\max}} |\langle b_u^r|\phi^{\mathrm{Q}}\rangle|^2$ and $E_{\min}$ and $E_{\max}$ are given by the following expressions:

$$\begin{aligned} E_{\min} &= 2P(b_u \notin I_{\min})^{1/2}\left(\frac{(1-p)\Delta_L(p)}{2(2L-1)I_B}\right)^{1/2} + \frac{(1-p)\Delta_L(p)}{2(2L-1)I_B} \\ E_{\max} &= 2P(b_u \in I_{\max})^{1/2}\left(\frac{(1-p)\Delta_L(p)}{2(2L-1)I_B}\right)^{1/2} + \frac{(1-p)\Delta_L(p)}{2(2L-1)I_B}. \end{aligned} \tag{45}$$

Notice that, given the degree of validity L of the classical sector initial data, we can play with the interval I_0 and with the values of I_B and p, which in turn impose a value on $\Delta_L(p)$, to minimize the error of the predictions for the probabilities. To obtain some feeling about the accuracy of the predictions let us choose some explicit values for L, I_B and p. Let $I_B = \delta_L(\hat{B})$ so that $\Delta_L(p) = 2I_B/(1-p)^{1/2L}$. The errors $E_{\min}$ and $E_{\max}$ become

$$\begin{aligned} E_{\min} &= 2P(b_u \notin I_{\min})^{1/2}\left(\frac{(1-p)^{\frac{2L-1}{2L}}}{2L-1}\right)^{1/2} + \frac{(1-p)^{\frac{2L-1}{2L}}}{2L-1} \\ E_{\max} &= 2P(b_u \in I_{\max})^{1/2}\left(\frac{(1-p)^{\frac{2L-1}{2L}}}{2L-1}\right)^{1/2} + \frac{(1-p)^{\frac{2L-1}{2L}}}{2L-1}. \end{aligned} \tag{46}$$

Let us consider $L = 1$, meaning that the classical sector initial data (O_i^0, δ_i) is first-order valid. Let us also choose $p = 0.99$. We can then state that, in the representation of $\hat{A}$, the states $|\xi_u\rangle$ (21) have, at least, 99% of their probability confined to the intervals $I_u = [b_u - 20\delta_1(\hat{B}), b_u + 20\delta_1(\hat{B})]$, where $\delta_1(\hat{B})$ is given by (24) and is of the size of a classical error margin. Moreover,

$$\begin{aligned} E_{\min} &= 2 \times 0.31 \times P(b_u \notin I_{\min})^{1/2} + 0.1 \\ E_{\max} &= 2 \times 0.31 \times P(b_u \in I_{\max})^{1/2} + 0.1 \end{aligned} \tag{47}$$

and thus, in the worst case,

$$P(b_u \in I_{\min}) - 0.72 \leqslant P(a_i \in I_0) \leqslant P(b_u \in I_{\max}) + 0.72 \tag{48}$$

with the difference between the ranges of $I_{\max}$, $I_{\min}$ and I_0 being given by $20\delta_1(\hat{B})$. An error of 72% is huge. The reason for such a large error lies in the fact that the conditions imposed over $|\phi^{\mathrm{c}}\rangle$ are the least restrictive possible, $L = 1$. In other words, the classical initial data are the least valid possible. To see what happens when we increase the degree of validity of the classical sector initial data let us make $L = 10$. We consider once again $I_B = \delta_{10}(\hat{B})$ but, this time, let us choose $p = 0.999\,99 = 1 - 10^{-5}$. We have $\Delta_{10}(p = 0.999\,99) = 3.6\delta_{10}(\hat{B})$ and

$$\begin{aligned} E_{\min} &= 0.0019P(b_u \notin I_{\min})^{1/2} + 9.4 \times 10^{-7} \\ E_{\max} &= 0.0019P(b_u \in I_{\max})^{1/2} + 9.4 \times 10^{-7} \end{aligned} \tag{49}$$

and thus, in the worst case, $P(b_u \in I_{\min}) - 0.0019 \leqslant P(a_i \in I_0) \leqslant P(b_u \in I_{\max}) + 0.0019$. This is an error of 0.19% with the difference between the ranges of $I_{\max}$, $I_{\min}$ and I_0 decreasing to $3.6\delta_{10}(\hat{B})$.

3. Half quantization

We start by pointing out, once again, that the predictions $P(a_i \in I_0)$ and its error margins can be obtained if we have the knowledge of the operator $\hat{B}$ and in no way require (except to obtain $\hat{B}$) the knowledge of the full quantum operator $\hat{A}$. This means that if we were able to calculate $\hat{B}$ directly then we would be able to make predictions for the evolution of the half quantum system without firstly having to obtain its fully quantum version. Therefore, the aim of this section is to obtain a framework able to provide the operator $\hat{B}$ directly from the initial data of the half quantum system without requiring previous knowledge of the full quantum theory.

3.1. The unquantization map

In section 2.2 we present the first definition of the unquantization map. The motivation for that definition of V_0 was the fact that it validates the expansions (19) and (20), which were crucial to develop the entire approximation procedure presented in the last section. It has already been pointed out that the map V_0 is a trivial generalization of the unquantization map presented in [15]. Let us name this last map V_0^{c}. In fact the action of V_0 over a classical sector operator is identical to the action of V_0^{c}. In that paper we saw that V_0^{c} is just the inverse map of the Dirac quantization [16, 17]. Taking this result into account we present a new, however equivalent, definition of the half unquantization:

Definition 2 (second unquantization map). *Let $\wedge$ be the Dirac quantization map [17], $\wedge : \mathcal{A}(T^*M_1) \longrightarrow \mathcal{A}(\mathcal{H}_1)$ and let $\hat{A} = \sum_j \hat{A}_j^{\mathrm{Q}} \hat{A}_j^{\mathrm{c}}$ where $\hat{A}_j^{\mathrm{Q}}$ and $\hat{A}_j^{\mathrm{c}}$ are arbitrary multiple products of quantum and classical sector operators, respectively (16). The unquantization $\vee$ is a map from the algebra $\mathcal{A}(\mathcal{H})$ to the algebra $\mathcal{S}$ (check for the definition of $\mathcal{S}$ in definition 1) defined by the following rules:*

(1) $\vee(\hat{A}) = \sum_j \vee(\hat{A}_j^{\mathrm{Q}}) \vee (\hat{A}_j^{\mathrm{c}})$.
(2) $\vee(\hat{A}_j^{\mathrm{c}}) = A_i^{\mathrm{c}}$ *iff* $\wedge(A_j^{\mathrm{c}}) = \hat{A}_j^{\mathrm{c}}$.
(3) $\vee(\hat{A}_j^{\mathrm{Q}}) = \hat{A}_j^{\mathrm{Q}}$.

Let us study some properties of $\vee$:

(1) The map $\vee$ is equivalent to the map V_0 of definition 1. This is so because rule (2) of the definition of $\vee$ is equivalent to rule (3) of the definition of V_0. This fact was extensively discussed in [15].

(2) Just like V_0, the map $\vee$ is beset by order problems. In general there are several different classical sector observables that when quantized yield the same quantum operator. Let $A_1^{\mathrm{c}} \neq A_2^{\mathrm{c}}$ be two such observables, i.e. $\wedge(A_1^{\mathrm{c}}) = \hat{A}^{\mathrm{c}}$ and also $\wedge(A_2^{\mathrm{c}}) = \hat{A}^{\mathrm{c}}$. This means that $\vee(\hat{A}^{\mathrm{c}}) = A_1^{\mathrm{c}}$ but also $\vee(\hat{A}^{\mathrm{c}}) = A_2^{\mathrm{c}}$. Hence the map $\vee$ is not one to one. On the other hand, the predictions (44) and (45) of the last section are made for a general quantum operator $\hat{A}$ (for instance $\hat{A} = \hat{A}^{\mathrm{c}}\hat{A}^{\mathrm{Q}}$) and might be obtained using any of the operators $\hat{B} = V_0(\hat{A})$ (or equivalently, $\hat{B} = \vee(\hat{A})$). Therefore, the ambiguity of $\vee$ could be problematic if the predictions obtained by using two different $\hat{B}$ (for instance $\hat{B}_1 = A_1^{\mathrm{c}}\hat{A}^{\mathrm{Q}}$ and $\hat{B}_2 = A_2^{\mathrm{c}}\hat{A}^{\mathrm{Q}}$) were inconsistent.

However, one can easily realize that this is not the case. The difference $\hat{B}_2 - \hat{B}_1$ is proportional to a leading factor of $c_j\hbar^2$ (where c_j is the highest numerical coefficient of $\hat{A}$ displayed in the orders from which $\hat{B}_1$ and $\hat{B}_2$ were calculated). We have already seen following (20) that the validity of the predictions of the last section rests upon the premise that the numerical factors of $\hat{B} \gg c_j\hbar^2$ (otherwise $\hat{B}$ cannot be considered for reproducing the predictions of $\hat{A}$). We also saw that this premise is satisfied if the original operator $\hat{A}$, from

which $\hat{B}$ was calculated, satisfies some order conditions. Therefore, and if $\hat{B}_2$ and $\hat{B}_1$ are both valid operators, obtained from $\hat{A}$ displayed in required orders, the difference $\hat{B}_2 - \hat{B}_1$ is not meaningful when compared to the imprecision (which is proportional to the numerical factors of $\hat{B}$ (24)) associated with the predictions obtained by using either $\hat{B}_1$ or $\hat{B}_2$. In conclusion, $\hat{B}_1$ and $\hat{B}_2$ provide *physical predictions* which are consistent with each other, solving the ambiguity.

(3) Unquantizing the product of two classical sector operators: let us consider two general classical observables B and C. To quantize BC one uses the symmetrization rule: $\wedge(BC) = 1/2(\hat{B}\hat{C} + \hat{C}\hat{B})$. We just use the same rule for the unquantization:

$$\vee(\hat{B}\hat{C}) = \vee\left(\frac{\hat{B}\hat{C} + \hat{C}\hat{B}}{2} + \frac{1}{2}[\hat{B}, \hat{C}]\right) = BC + \frac{1}{2}\mathrm{i}\hbar\{B, C\}. \tag{50}$$

Notice that the prescription is beset by order problems (comment (2)).

(4) Unquantization of a self-adjoint operator: if $\hat{A} = \hat{A}^{\mathrm{c}}$ then we obtain from rule (2) $\vee(\hat{A}^{\dagger}) = \vee(\hat{A})^*$. For the case of a general operator $\hat{A} = \sum_j \hat{A}^{\mathrm{c}}_j \hat{A}^{\mathrm{Q}}_j$ we have

$$\vee(\hat{A}^{\dagger}) = \sum_j \vee(\hat{A}^{\mathrm{c}}_j)^*(\hat{A}^{\mathrm{Q}}_j)^{\dagger} = \vee(\hat{A})^{\dagger} \tag{51}$$

and if $\hat{A} = \hat{A}^{\dagger}$ then $\vee(\hat{A})^{\dagger} = \vee(\hat{A}^{\dagger}) = \vee(\hat{A})$ and so $\vee(\hat{A})$ is also self-adjoint.

(5) Unquantizing the brackets: for the simplest case of $\hat{A} = \hat{A}^{\mathrm{c}}$ and $\hat{B} = \hat{B}^{\mathrm{c}}$, from rule (2) one immediately has

$$\vee[\hat{A}, \hat{B}] = \vee(\wedge(\mathrm{i}\hbar\{A, B\})) = \mathrm{i}\hbar\{A, B\}. \tag{52}$$

For the most general case let us first put $\hat{A} = \hat{A}^{c}\hat{A}^{\mathrm{Q}}$ and $\hat{B} = \hat{B}^{\mathrm{c}}\hat{B}^{\mathrm{Q}}$, which only excludes sums of operators which, using rule (1), are straightforward to handle. We obtain

$$\begin{aligned}[\hat{A}, \hat{B}] &= \hat{A}^{\mathrm{c}}\hat{B}^{\mathrm{c}}[\hat{A}^{\mathrm{Q}}, \hat{B}^{\mathrm{Q}}] + [\hat{A}^{\mathrm{c}}, \hat{B}^{\mathrm{c}}]\hat{B}^{\mathrm{Q}}\hat{A}^{\mathrm{Q}} \\ &\Longrightarrow \vee([\hat{A}, \hat{B}]) = \vee(\hat{A}^{\mathrm{c}}\hat{B}^{\mathrm{c}})[\hat{A}^{\mathrm{Q}}, \hat{B}^{\mathrm{Q}}] + \vee([\hat{A}^{\mathrm{c}}, \hat{B}^{\mathrm{c}}])\hat{B}^{\mathrm{Q}}\hat{A}^{\mathrm{Q}}\end{aligned} \tag{53}$$

and using (50) and (52) we obtain

$$\vee([\hat{A}, \hat{B}]) = A^{\mathrm{c}}B^{\mathrm{c}}[\hat{A}^{\mathrm{Q}}, \hat{B}^{\mathrm{Q}}] + \frac{\mathrm{i}\hbar}{2}\{A^{\mathrm{c}}, B^{\mathrm{c}}\}(\hat{A}^{\mathrm{Q}}\hat{B}^{\mathrm{Q}} + \hat{B}^{\mathrm{Q}}\hat{A}^{\mathrm{Q}}). \tag{54}$$

3.2. Half quantization and half quantum mechanics

Equation (54) can be displayed in a slightly different form:

$$\vee([\hat{A}, \hat{B}]) = [\vee(\hat{A}), \vee(\hat{B})] + \mathrm{i}\hbar\{\{\vee(\hat{A}), \vee(\hat{B})\}\} = (\tilde{A}, \tilde{B}) \tag{55}$$

where the double brackets are defined by

$$\begin{aligned}\{\{\vee(\hat{A}), \vee(\hat{B})\}\} &= \tfrac{1}{2}\{A^{\mathrm{c}}, B^{\mathrm{c}}\}(\hat{A}^{\mathrm{Q}}\hat{B}^{\mathrm{Q}} + \hat{B}^{\mathrm{Q}}\hat{A}^{\mathrm{Q}}) \\ &= \frac{1}{2}\sum_i \frac{\partial\tilde{A}}{\partial q_i}\frac{\partial\tilde{B}}{\partial p_i} - \frac{\partial\tilde{A}}{\partial p_i}\frac{\partial\tilde{B}}{\partial q_i} + \frac{\partial\tilde{B}}{\partial p_i}\frac{\partial\tilde{A}}{\partial q_i} - \frac{\partial\tilde{B}}{\partial q_i}\frac{\partial\tilde{A}}{\partial p_i}\end{aligned} \tag{56}$$

and we have introduced the notation $\tilde{A} = \vee(\hat{A})$ and defined the new bracket $(\,,\,) = [\,,\,] + \mathrm{i}\hbar\{\{\,,\,\}\}$. This bracket was first proposed in [3,4]. The motivation to define it this way was given in terms of the properties of the emerging theory, namely that it properly generalizes both quantum and classical mechanics. The bracket is known to be antisymmetric and multilinear but it does not satisfy the Jacobi identity. This caused much debate in the literature [1, 2, 19–21]. We will come back to this problem in the conclusions. Firstly let us finish the presentation of the dynamical structure of half quantum mechanics.

Given the unquantization map of definition 2 we are able to define a quantization prescription mapping a general classical dynamical system to the corresponding half quantum one. One just needs to specify the classical and the quantum sectors to be of the original classical theory, that is to provide T^*M_1 and T^*M_2. Let then $\mathcal{F}$ be the algebra of observables over $T^*M = T^*M_1 \otimes T^*M_2$. The remaining notation is in accordance with the previous definition.

Definition 3 (Half quantization map). *The half quantization is defined to be the map*

$$\cap : \mathcal{F} \longrightarrow \mathcal{S} \qquad \cap = \vee \circ \wedge \qquad A \longrightarrow \tilde{A} = \cap(A) \tag{57}$$

where $\wedge : \mathcal{F} \longrightarrow \mathcal{A}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ *is the Dirac quantization map and* $\vee$ *is the unquantization map of definition 2.*

The properties of $\bigcap$ follow directly from its definition: let $A, B \in \mathcal{F}$:

(1) $\cap(f(q_i, p_i)) = f(q_i, p_i)\hat{I}$ and $\cap(q_\alpha) = \hat{q}_\alpha$, $\cap(p_\alpha) = \hat{p}_\alpha$, $i = 1 \ldots M$, $\alpha = (M+1) \ldots (M+N)$.
(2) $\cap$ is a linear map: $\cap(A + bB) = \cap(A) + b \cap (B)$, $b \in \mathcal{C}$.
(3) $\cap(f(q_i, p_i)g(q_\alpha, p_\alpha)) = f(q_i, p_i) \wedge (g(q_\alpha, p_\alpha))$.
(4) $\cap(\{A, B\}) = (\mathrm{i}\hbar)^{-1}(\cap(A), \cap(B))$.

We are now in position to study the theory resulting from applying the half quantization procedure to a given classical theory. First we have to choose a CSCO for the quantum sector of the theory. Let it be, for instance, the set $\{\hat{q}_\alpha\}, \alpha = M+1 \ldots M+N$. The initial data for the classical sector is given by the set $\{q_i^0 = q_i(t_0), p_i^0 = p_i(t_0)\}$ and the corresponding error margins δ_{q_i} and $\delta_{p_i}, i = 1 \ldots M$. The quantum initial data are given by the initial data wavefunction $|\phi^{\mathrm{Q}}\rangle \in \mathcal{H}_2$. The dynamical evolution of the half quantum system is determined by the following set of equations:

$$\dot{\tilde{O}}_k = \cap(\{O_k, H\}) = \frac{1}{\mathrm{i}\hbar}(\tilde{O}_k, \tilde{H}) \tag{58}$$

where $\tilde{O}_k, k = 1 \ldots 2(M+N)$ is any of the fundamental variables. The former set of equations has the formal solutions

$$\tilde{O}_k(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t}{\mathrm{i}\hbar}\right)^n (\ldots(\tilde{O}_k, \tilde{H}), \ldots, \tilde{H}) = O_k(\hat{q}_\alpha, \hat{p}_\alpha, q_i^0, p_i^0, t). \tag{59}$$

Notice that (58) is just the same set of equations as that resulting from applying the unquantization map to the standard quantum evolution equations for the observables $\hat{O}_k(t)$ and so is the solution (59). Hence, the observables $\tilde{O}_k(t)$ are just the operators $\hat{B}$ we need to supply to obtain the predictions (44) and (45).

4. Example

To illustrate the procedure by which half quantum mechanics makes predictions for the time evolution of a given dynamical system, let us consider the following system of two interacting particles, described by the Hamiltonian

$$\tilde{H} = \frac{\hat{P}^2}{2M} + \frac{p^2}{2m} + kq\hat{P} \tag{60}$$

where $(\hat{Q}, \hat{P})$ are the fundamental observables of the quantum particle of mass M, (q, p) are the canonical variables of the classical particle of mass m and k is a coupling constant. The

initial data for the quantum particle are given by $|\phi^{Q}\rangle$, while the initial-time configuration of the classical particle is described by the data $\{q(0), p(0), \delta_q(0), \delta_p(0)\}$.

Solving the half quantum equations of motion (58) we obtain the time evolution of the fundamental observables of the half quantum system

$$\begin{aligned}
&\tilde{q}(t) = q(0) + \frac{p(0)}{m}t - \frac{k\hat{P}(0)}{2m}t^2 \\
&\tilde{p}(t) = p(0) - k\hat{P}(0)t \\
&\tilde{Q}(t) = \hat{Q}(0) + \left(\frac{\hat{P}(0)}{M} + kq(0)\right)t + \frac{k}{2m}p(0)t^2 - \frac{k^2}{6m}\hat{P}(0)t^3 \\
&\tilde{P}(t) = \hat{P}(0)
\end{aligned} \tag{61a}$$

together with the errors $\delta_L(\hat{B})$ of the half quantum operators (24)

$$\begin{aligned}
&\delta_L(\tilde{q}(t)) = \delta_q(0) + \left|\frac{t}{m}\right|\delta_p(0) \\
&\delta_L(\tilde{p}(t)) = \delta_p(0) \\
&\delta_L(\tilde{Q}(t)) = |kt|\delta_q(0) + \left|\frac{kt^2}{2m}\right|\delta_p(0) \\
&\delta_L(\tilde{P}(t)) = 0
\end{aligned} \tag{61b}$$

a result that is valid for all $L \in \mathcal{N}$. The spreads are of the general form $\Delta_L(\hat{B}) = \delta_L(\hat{B})/(1-p)^{1/2L}$, for $\hat{B} = \tilde{q}(t)$, $\tilde{p}(t)$, $\tilde{Q}(t)$ or $\tilde{P}(t)$.

Let $\hat{A}$ be one of the full quantum operators $\hat{q}(t)$, $\hat{p}(t)$, $\hat{Q}(t)$ or $\hat{P}(t)$, the ones we would have obtained if we had performed the full quantum treatment of the system with Hamiltonian $\hat{H} = \hat{P}^2/2M + \hat{p}^2/2m + k\hat{q}\hat{P}$. Let $\hat{B}$ be the corresponding half quantum operator (61). Moreover, let $|a_i^n\rangle$ be a complete set of eigenstates of $\hat{A}$ (a_i is the associated eigenvalue and n is the degeneracy index) and $|b_u^r\rangle$ a complete set of eigenvectors of $\hat{B}$ (b_u is the associated eigenvalue and r is the degeneracy index). If the classical sector initial data are taken to be first-order valid ($L = 1$) the half quantum predictions for the outputs of a measurement of the fully quantum operator $\hat{A}$ (choosing $p = 0.99$ and $I_B = \delta_1(\hat{B})$) are given by (48)

$$P(b_u \in I_{\min}) - 0.72 \leqslant P(a_i \in I_0) \leqslant P(b_u \in I_{\max}) + 0.72$$

where $I_0 = [a^0 - D, a^0 + D]$ is an arbitrary interval centred at $a^0 \in \mathcal{R}$ with range $D \geqslant 20\delta_1(\hat{B})$, the error $\delta_1(\hat{B})$ is given by (61), $I_{\max} = [a^0 - (D + 20\delta_1(\hat{B})), a^0 + (D + 20\delta_1(\hat{B}))]$ and $I_{\min} = [a^0 - (D - 20\delta_1(\hat{B})), a^0 + (D - 20\delta_1(\hat{B}))]$. Moreover, $P(b_u \in I_{\min,\max}) = \sum_{r, b_u \in I_{\min,\max}} |\langle\phi^{Q}|b_u^r\rangle|^2$.

If the classical sector initial data $\{q(0), p(0), \delta_q(0), \delta_p(0)\}$ are tenth-order valid then, as we have seen, the precision of the half quantum predictions increases considerably (let $p = 0.999\,99$ and $I_B = \delta_{10}(\hat{B})$):

$$P(b_u \in I_{\min}) - 0.0019 \leqslant P(a_i \in I_0) \leqslant P(b_u \in I_{\max}) + 0.0019$$

where, this time, $I_{\max} = [a^0 - (D + 3.6\delta_{10}(\hat{B})), a^0 + (D + 3.6\delta_{10}(\hat{B}))]$, $I_{\min} = [a^0 - (D - 3.6\delta_{10}(\hat{B})), a^0 + (D - 3.6\delta_{10}(\hat{B}))]$ and $\delta_{10}(\hat{B}) = \delta_1(\hat{B})$ is given by (61).

Clearly the former predictions are not valid in general. They are valid if the two descriptions of the classical sector initial-time configuration, the classical $\{q(0), p(0), \delta_q(0), \delta_p(0)\}$ and the quantum $|\phi^{c}\rangle$, satisfy some consistency conditions. These conditions are the L-order classicality conditions. Let us just notice that, following the procedure of section 2.1, the $L = 1$ fundamental sequences (1) for this system are $S_1 = q$ and $S_2 = p$, which are exactly the same sequences as for the example of the harmonic

oscillator of section 2.1. Therefore the L-order classicality conditions for $|\phi^{c}\rangle$ are exactly the conditions (6), (7). As we saw, Gaussian wavefunctions provide well known solutions for which the classicality conditions reduce to (9). If, at the initial time, the classical sector is in such a state then the L-order half quantum predictions are valid.

5. Conclusions

The general prescription to derive a theory of coupled classical–quantum dynamics presented in this paper might be summarized in three main steps. (1) Identification of the properties that should be satisfied by the full quantum initial data so that it might be properly described by a set of half quantum initial data (section 2, equation (3)). (2) Establishment of a relation between a general full quantum observable and the corresponding half quantum one so that one is able to reproduce the predictions of quantum mechanics using the half quantum operators (equation (20)). This involves the derivation of a relation between the (central) eigenvectors of $\hat{B}$ and the eigenvectors of $\hat{A}$ (equation (23)) and then of a relation between the probabilities in the representation of $\hat{B}$ and of $\hat{A}$ (equations (44) and (45)). (3) Finally, the derivation of a framework providing the half quantum operators without requiring previous knowledge of the full quantum theory (section 3).

Certainly, there are many different ways of implementing this general plan (see for instance [2, 18]). In this paper we presented a particular derivation of a theory of coupled classical–quantum dynamics that was named half quantum mechanics. This theory, in the form of a set of axioms, was first presented in [3, 4] and its properties have been extensively discussed in the literature [1, 2, 19–21]. In particular, the fact that the bracket structure does not satisfy the Jacobi identity is known to be problematic, the dynamical structure displaying a set of undesirable properties (it is not unitary and time evolution does not preserve the bracket structure, just to mention two of the most intriguing). However, and despite the fact that the internal structure of half quantum mechanics is not the most desirable, the theory was shown to provide a valid description of coupled classical–quantum dynamics in the sense that it reproduces the results of quantum mechanics in the appropriated limit. The key issue in half quantum mechanics is, of course, the way in which its predictions should be interpreted. Associated with every prediction is an error margin, and within this error margin the theory is physically valid.

To finish we would like to make a few comments:

(a) There is an uncertainty associated with all predictions made by the half quantum theory. Since we do not have a complete knowledge of the initial data wavefunction we could not expect to have a complete deterministic prediction, much the same as what happens in classical mechanics. As expected, the degree of precision of the half quantum predictions is related to the classicality conditions that are assumed to be satisfied by the classical sector initial data wavefunction or, in other words, to the degree of validity of the classical initial data.

(b) A different bracket for classical–quantum dynamics has been presented in the literature [1]. The new theory was also postulated and motivated in terms of its properties. This has caused much debate over which would be the best structure for a theory of coupled classical–quantum dynamics. We would like to point out that Anderson theory might also be obtained through a procedure similar to that presented in this paper. To do this we just have to use a slightly different unquantization map. The deductive approach will provide a way of comparing the two theories with respect to their consistency with the full quantum description.

(c) Lastly, as a side result, we realized that the fact that the half quantum bracket does not satisfy the Jacobi identity is clearly a consequence of the fact that the unquantization map is not one to one. This might point out a path to obtain a new, better behaved theory of coupled classical–quantum dynamics [18].

Appendix. Error ket framework

The aim of this appendix is just to present some of the results of the error ket framework. For a more detailed presentation the reader should refer to [15]. Let us start by introducing the relevant definitions. Let $\hat{X}_i$, $i = 1 \dots n$ be a set of n operators acting on the Hilbert space $\mathcal{H}$ and let $|\psi\rangle$ be the wavefunction describing the system.

Definition (error ket). *We define the n-order mixed error ket* $|E(\hat{X}_1, \hat{X}_2, \dots, \hat{X}_n, \psi, x_1^0, x_2^0, \dots, x_n^0)\rangle$, *as the quantity*

$$|E(\hat{X}_1, \hat{X}_2, \dots, \hat{X}_n, \psi, x_1^0, x_2^0, \dots, x_n^0)\rangle = (\hat{X}_1 - x_1^0)(\hat{X}_2 - x_2^0) \dots (\hat{X}_n - x_n^0)|\psi\rangle \tag{62}$$

where x_i^0 *are complex numbers and the operators* $\hat{X}_i$ *do not need to be self-adjoint. The error bra* $\langle E(\hat{X}_1 \dots \hat{X}_n, \psi, x_1^0, \dots, x_n^0)|$ *is defined according to the definition of the error ket. When there is no risk of confusion we will also use the notation* $|E_{X_1,\dots,X_n}\rangle$ *for the mixed error ket. Moreover, when* $\hat{X}_1 = \hat{X}_2 = \cdots \hat{X}_n = \hat{X}$ *the error (62) is named 'n-order error ket' and we write it in the form* $|E_X^n\rangle = |E^n(\hat{X}, \psi, x^0)\rangle$.

Let us present some properties of the former quantity:

(a) The error ket provides a confinement of the wavefunction. Let $\hat{X}$ be self-adjoint and $x^0 \in \mathcal{R}$. Given $\langle E_X^n | E_X^n\rangle$, with each 'quantity of probability' p we can associate an interval I around x^0, $I = [x^0 - \Delta_n, x^0 + \Delta_n]$, such that the probability of obtaining a value $x \in I$ from a measurement of $\hat{X}$ is at least p. The size of the interval I is dependent on $\hat{X}$, ψ and x^0 only through the value of $\langle E_X^n | E_X^n\rangle$. We call the quantity $\Delta_n = \Delta_n(\hat{X}, \psi, x^0, p)$ the n-order spread of the wavefunction. Δ_n is given by

$$\Delta_n(\hat{X}, \psi, x^0, p) = \left(\frac{\langle E_X^n | E_X^n\rangle}{1-p}\right)^{1/2n}. \tag{63}$$

If $\hat{X}$ is not self-adjoint then the former result can also be obtained, but in this case I is a ball of radius Δ_n in the complex plane.

(b) Let $\hat{X}$ be self-adjoint and $x^0 \in \mathcal{R}$. The former result can be restated in the following way: given $\langle E_X^n | E_X^n\rangle$ and a distance d, the probability of obtaining a value $x \notin [x^0 - d, x^0 + d]$ from a measurement $\hat{X}$ is at the most $\langle E_X^n | E_X^n\rangle / d^{2n}$. In fact (let $|x, k\rangle$ be a complete set of eigenvectores of $\hat{X}$, where x is the associated eigenvalue and k is the degeneracy index),

$$\begin{aligned}\langle E_X^n | E_X^n\rangle &= \sum_{x,k} (x - x^0)^{2n} |\langle\psi|x, k\rangle|^2 \\ &\geqslant \sum_{x \notin [x^0-d, x^0+d], k} (x - x^0)^{2n} |\langle\psi|x, k\rangle|^2 \\ &\geqslant d^{2n} \sum_{x \notin [x^0-d, x^0+d], k} |\langle\psi|x, k\rangle|^2 \end{aligned} \tag{64}$$

and this implies

$$\sum_{x \notin [x^0-d, x^0+d], k} |\langle\psi|x, k\rangle|^2 \leqslant \frac{\langle E_X^n | E_X^n\rangle}{d^{2n}} = \frac{\Delta_n(p)^{2n}(1-p)}{d^{2n}}. \tag{65}$$

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